Micro I. Dr. Wilcox. Problem set #9.

(1) This question illustrates one of many ways of thinking about would-be violators of rules, laws, orders, etc. and what may be done to them. These become the basis of thinking about questions of crime, fraud, malfeasance and so forth.

A driver with wealth w is deciding whether to park legally or illegally. If she parks legally, she will simply have her wealth w for sure. However, if she parks illegally, two things change. She will save time that has a dollar value s to her, and she adds this s to her wealth w, regardless of whether or not she gets a ticket when parked illegally. But she will get a ticket with probability p; and if this happens, she must pay a fine equal to f dollars. She has a strictly concave elementary utility function for money that is everywhere continuous and twice differentiable, and she acts as an expected utility maximizer.

Questions:

(a) The legal/illegal parking decision is being conceived of as a "decision under risk" problem here. Parking legally is simply the "sure thing" lottery of getting w with certainty. Write down the "risky" lottery associated with illegal parking. Express both legal and illegal parking as two alternative decision trees representing these two lotteries that the driver can choose.

(b) If the driver is even going to <u>consider</u> parking legally, what has to be true of the relationship between s and f?

(c) Define $S(p,f)$ as the exact dollar value of time savings that would make a driver with utility of money $u(z)$ indifferent between the legal and illegal parking lotteries. (Implicitly, that function would depend on what function $u(z)$ a driver has, but you may "suppress this argument" as I am doing here for simplicity). Write down the mathematical identity that defines the function $S(p,f)$. Explain in words how this mathematical definition of $S(p,f)$ implies the following "decision rule" for our driver:

If $S(p,f) > s$, park legally.
If $S(p,f) < s$, park illegally.
If $S(p,f) = s$, do either one.

(d) Use formal comparative statics on the identity you wrote down in (c) to find out how changes in p and f affect S. Sign these expressions. Interpret the signs from a behavioral viewpoint. (Hint: Think of a situation where s has some distribution in the population of drivers, and assume that $u(z)$ does not vary over drivers. Actually, you don't need that latter assumption but it does make talking about this a bit simpler).

(e) Show that the elasticity of S with respect to f exceeds the elasticity of S with respect to p. (Hint: Start by bounding $\partial S/\partial p$ from above, using a second-order approximation of it around an appropriately chosen point and the fact that $u(z)$ is strictly concave. There are other ways to proceed, too.)

(f) <u>Rewrite</u> this problem so that it applies to any decision to break any rule of any kind. Do not change the notation—just generalize the meanings of w, s, p and f as needed, and certain words, so that the problem statement is about any decision to break any rule.

(2) In this problem, you are going to work with lotteries that have a continuum of possible outcomes. In such cases, remember, we represent lotteries as probability density functions over possible final amounts of wealth or money z, that is as $f(z)$ or $g(z)$. Notice that since we are assuming that these lotteries are already expressed as distributions over final wealth, we do not have to explicitly introduce a current wealth level w to properly talk about the expected utility of these lotteries.

From time to time, researchers in many areas, from finance and economics to psychology, have used (or tested) "mean-variance analysis" as a way to characterize the value of a lottery. Mean-variance analysis presumes that we can write a consumer's valuation V of any lottery $f(z)$ as some function of just the lottery's mean μ_f and standard deviation σ_f. That is, we assume that we can define:

$$V[f] \equiv V(\mu_f, \sigma_f).$$

The important point here is that value can be expressed so that it only depends on the mean and standard deviation of a lottery and nothing else about that particular lottery, such as higher order moments like skew. Notice that this doesn't imply that V is the same for every decision maker; the form of V might well vary between decision makers.

The purpose of this exercise is to specify some conditions under which mean-variance analysis is an implication of expected utility theory—that is, under what conditions would it be ok to describe an expected utility maximizer by a mean-variance analysis. In the case of continuous p.d.f. lotteries like those in this problem, recall that expected utility theory has this representation theorem:

$$\text{for any decision maker, } \exists\ u(z) \mid f(z) \succ g(z) \Leftrightarrow \int u(z) f(z) dz > \int u(z) g(z) dz$$

(a) Suppose that an expected utility-maximizing individual has a quadratic utility function $u(z) = \alpha z + \beta z^2$, where $\alpha > 0$ and $\beta < 0$. You may assume that all outcomes that have positive probability density are in the interval [0,B] and that $\alpha + 2\beta B > 0$ (explain why you need to make this assumption for a quadratic utility of money function). Show that there exists a function $V(\mu_f, \sigma_f)$ of the kind described above such that:

$$f(z) \succ g(z) \Leftrightarrow V(m_f, s_f) > V(m_g, s_g)$$

Note: Showing the existence of such a function is equivalent to showing that $\int u(z)f(z)dz \equiv V(\mu_f, \sigma_f)$. Remember, the form of this function may well depend on the individual's utility function for money. What you are trying to show is that the function is completely independent of all information about the lottery considered, except for its mean and standard deviation--if we already know that $u(z) = \alpha z + \beta z^2$.

(b) Suppose instead now that you don't know the exact form of the individual's $u(z)$ anymore. Instead, the prior knowledge you have is about the parametric form of all the lotteries faced by the decision makers. Suppose in particular that you know that f(z) and g(z) are Normal probability density functions—that is, that lotteries are all Normally distributed random variables. Show that once again, there exists a function $V(\mu_f,\sigma_f)$ of the kind described above such that:

$$f(z) \succ g(z) \Leftrightarrow V(m_f, s_f) > V(m_g, s_g)$$

Notes: The same notes apply here. Realize that in part (b), the function *V* will not look like the function *V* you found in part (a). Some hints: (1) You will need an <u>infinite</u> Taylor expansion of u(z) around a well-chosen value of z (you may assume that *u*(*z*) is sufficiently well-behaved so that this infinite expansion perfectly approximates the function at all argument values, which is not true for arbitrary functions); and (2) You will need to use these facts about the central moments of Normally distributed random variables *z* having mean μ and standard deviation σ (consult a statistics book):

$E[z] = \mu$; $E[(z-\mu)^2] = \sigma^2$; $E[(z-\mu)^j] = 0$ for all odd *j*; and

$E[(z-\mu)^j] = a_j\sigma^j$ for all even *j*, where a_j is a coefficient whose values depends only on j (you can find the exact form of those coefficients in a statistics book but it is not necessary to do so in order to answer this question).