Dr. Wilcox. Micro I. Problem set #7.

(1) Consider the utility function for x_1 and x_2 given by:

$$U(x_1, x_2) = (x_1^r + x_2^r)/r$$

This is called the "constant elasticity of substitution" or CES utility function because the quantity $\sigma = 1/(1-\rho)$ is equal to the so-called "elasticity of substitution" of this class of utility functions, which is constant regardless of the values of x_1 and x_2. Not surprisingly, the elasticity of substitution is a measure of how substitutable x_1 and x_2 are. Look up "elasticity of substitution" in Deaton and Muellbauer or almost any graduate micro text and review the concept (if the discussion is carried on in terms of production functions, rather than utility functions, don't worry about that--it applies to both).

The CES utility function contains the following three utility functions as special cases:

$\rho = 1 \Rightarrow U(x_1,x_2) = x_1 + x_2$, the "linear utility function." x_1 and x_2 are called "perfect substitutes" in this case and the elasticity of substitution σ is infinite.

$\lim(\rho \to 0)$ yields $U(x_1,x_2) = ln(x_1) + ln(x_2)$, which is a monotonic transformation of the Cobb-Douglas utility function with $\alpha = \beta$. x_1 and x_2 are moderately substitutable in this case, with the elasticity of substitution $\sigma = 1$.

$\lim(\rho \to -\infty)$ yields $U(x_1,x_2) = min(x_1,x_2)$ (after a monotone increasing transformation) which is the "Leontief utility function." x_1 and x_2 are wholly unsubstitutable in this "perfect complements" case, with the elasticity of substitution $\sigma = 0$.

Comment: Normally, if we are talking about both production functions and utility functions at the same time, the CES functional form would be written in this way:

$$U(x_1, x_2) = (x_1^r + x_2^r)^{1/r} .$$

The way I first wrote it above is just a monotone transformation of this version for any $\rho \neq 0$; it is just a little easier to demonstrate the limiting cases mathematically if you start with my form rather than the more usual one I just gave you here. This more usual way of writing the function is the way it would always be written as a production function, but in the case of utility functions, it doesn't really matter which form you look at.

Anyway, on with the questions. But here is one overarching hint: Draw pictures and think about intuition to get you going. The general issue you are looking at in this problem set is the relationship between degree of substitutability on the one hand, and the behavior of price indices and consumer surplus measures on the other. If you begin this problem by thinking about that issue in a general way, the questions below will make good sense.

(a) Let p_1 and p_2 be the prices of x_1 and x_2, and suppose the consumer has income M. For each of three special case utility functions above (the linear, Cobb-Douglass and Leontief, all special cases of the CES), derive the Marshallian demands, Hicksian demands, indirect utility function, and expenditure function. (Hint: Don't try to get the linear or Leontief demands by using the method of LaGrange—it won't work, since these utility functions either take you to corner solutions most of the time, or to a place where the marginal rate of substitution doesn't exist. Instead, reason it out graphically. Very big hint: Your last problem set will be helpful.)

(b) Suppose that $(p_1,p_2) = (1,1)$ and $M = 2$ in some "initial" state of affairs. Then, suppose income and the price of good 2 stay the same, but that the price of good 1 rises to some number $\theta > 1$, to produce some "final" state of affairs. For each of the three utility functions above:

(i) Calculate the change in the consumption of goods 1 and 2 for each of the three utility functions. (Note: In the case of the linear utility function, this will be indeterminate. Explain why. Then assume throughout this problem that at prices $(p_1,p_2) = (1,1)$, the consumer with the linear utility function consumes the bundle $(x_1,x_2) = (1,1)$, explaining why one might regard this as a reasonable "average consumption bundle" for the consumer with the linear utility function).

(ii) Calculate the "compensating variation" or CV—the minimum amount of extra income the consumer would need to receive in order to remain as well off at $p_1 = \theta$ as she was at $p_1 = 1$—for each of the three utility functions. Order these from highest to lowest for the three utility functions and, from that ordering, draw an inductive conclusion about the relationship between elasticity of substitution and CV.

(iii) Now, calculate the "equivalent variation" or EV—the maximum amount the consumer would be willing to pay in order to change the "final" price θ of good 1 back to the "initial" price 1 of good 1. Order these from highest to lowest, and draw a conclusion about the relationship between elasticity of substitution and EV.

(iv) Verify, for each utility function, that $CV \geq EV \geq 0$.

(v) Evaluate the expression $(CV - EV)/EV$ (regard this as equal to zero if the denominator and numerator are both equal to zero), the proportional discrepancy between CV and EV, for each utility function. Order these from highest to lowest, and draw a conclusion about the proportional discrepancy between CV and EV and the elasticity of substitution.

(c) Calculate the Laspeyres and Paasche cost-of-living indices for each of the three utility functions. Also calculate the true cost of living indices using both initial utility and final utility as the "reference utility level." Do the following for each utility function:

(i) Verify that the two true cost-of-living indices (that is, the "initial period" index and the "final period index") are equal to each other for each of the three utility functions. As I indicated in class, this is in fact true for any homothetic utility function: So, before doing anything else, prove that the CES utility function is homothetic—establishing that what you just found is sensible. Since the two are equivalent here, I'll stop referring to the "two" true indices and just say "the true index" from here on out.

(ii) Order the true cost-of-living index for the three utility functions from lowest to highest, and draw a conclusion concerning the relationship between the elasticity of substitution and the change in the cost of living due to a price increase. Relate this result to what you showed in (b-ii) and (b-iii) above.

(iii) Verify that the Laspeyres index exceeds or equals the true index for all three utility function. If it should happen that the two are equal for one of the utility functions, explain why that happens with a graph. Use Deaton and Muellbauer’s figures to get started, and just substitute the relevant indifference curve type and change the figures accordingly.

(iv) Verify that the true index exceeds or equals the Paasche index for all three utility functions. If it should happen that the two are equal for one of the utility functions, explain why that happens with a graph. Use Deaton and Muellbauer’s figures to get started, and just substitute the relevant indifference curve type and change the figures accordingly.