

Dr. Wilcox. Micro I. Problem Set #4.

1-a) (From Nicholson, page 393, problem 13.6; a price discrimination problem.) Universal Widget produces high-quality widgets at its plant in Gulch, Nevada, for sale throughout the world. The widget cost function for  $Q$  total widgets is  $C(Q) = Q^2/4$ . Widgets are demanded in Australia and Lapland, where the market demand curves for widgets are  $q^a = 100 - 2p^a$  and  $q^l = 100 - 4p^l$ , respectively, where  $p^a$  and  $p^l$  are the prices Universal Widget sets in Australia and Lapland, respectively. Assuming that Universal Widget can prevent resale of widgets between the Australian and Lapland widget markets, and so can set different prices in the two markets, how many widgets should it sell (and at what price in each market) in order to maximize total profits?

1-b) (From Nicholson, pages 539-40, problem 17.4) General Widget or GW has a monopoly on widget production and, moreover, is the only hirer of widget makers in the world. Suppose that the production function for widgets is  $Q = 2L$ , where  $L$  is the number of widget makers that GW hires. (You will notice that there is no capital in this problem: You should therefore assume that widget manufacturing either requires no capital, or that this is a short-run production function--i.e., capital stock is fixed and not subject to choice--and you are being asked to find short-run optima--i.e., you need only pick the right amount of the variable input labor.) Suppose that the world inverse demand for widgets is  $p = 100 - q$ , and that widget makers competitively supply their labor according to the inverse supply function  $w = 20 + 2L$ . How many widgets should GW produce in order to maximize profits, and at that output level, what will  $L$ ,  $w$  and  $p$  be?

1-c) (from Nicholson, p. 607, problem 19.4). Let  $z$  denote a firm's total "product differentiation" (or advertising) costs and let  $g(Q)$  denote its production and distribution costs, so that its total cost of making and selling  $Q$  units of output is given by  $C(Q) = z + g(Q)$ . Suppose that the demand for its product is given by  $q = q(p, z)$ , where  $p$  is the price it charges for the product. Let  $e_{q,z}$  and  $e_{q,p}$  denote the elasticity of this demand function with respect to its product differentiation (advertising) expenditures  $z$  and the price  $p$  the firm charges, respectively. (Note...when a demand function has several arguments, we can define elasticities with respect to any argument we like in the usual way--just use partial derivatives instead of ordinary derivatives.) Prove that for a profit maximizing firm, the following relationship must be true:  $z/pq = e_{q,z}/e_{q,p}$  (that is, that the ratio of advertising expenditures to total sales revenue is equal to this ratio of elasticities). Hint...this problem is easiest set up as the problem of choosing  $z$  and  $p$  (not choosing  $z$  and  $q$ ). When a monopolist's demand function can be inverted, you can always either think of the monopolist as choosing  $q$  (from which a price will be determined by the inverse demand function) or as choosing  $p$  (from which a quantity will be determined by the ordinary demand function). Which way is most convenient depends on the problem to be solved and what you are trying to show.

1-d) (from Nicholson, p. 662, problem 21.10). The town of Podunk has decided to provide security services to its residents by hiring workers  $L$  and guard dogs  $D$  per period. Security services  $S$  per period are produced according to the function  $S = (LD)^{0.5}$ , and residents of the town wish to consume 10 units of such services per period. Suppose at first that both  $L$  and  $D$  rent competitively at prices of \$1.00 per period each. How much of each input should Podunk hire to minimize the costs of providing 10 units of security services? Now suppose instead that

Podunk is the only hirer of security workers  $L$ , and that they are supplied according to the supply function  $L = 10w$ , where  $w$  is the price that Podunk pays these workers. In this case, how much of each input will Podunk hire in order to minimize its cost of providing 10 units of security services, and what wage will it pay security workers?

(2) A dictator wants to produce exactly  $Q^T$  units of a good. He has two plants, plant 1 and plant 2, in which to do this. The cost of producing  $Q_1$  units in plant 1 is  $C_1(Q_1)$  and the cost of producing  $Q_2$  units in plant 2 is  $C_2(Q_2)$ . The dictator wants to choose  $Q_1$  and  $Q_2$  so as to minimize the total cost of producing the total  $Q^T$  units of output. Assume that the cost functions at each plant are twice differentiable, strictly increasing and strictly convex. Write down the dictator's objective function and constraint, and because this constraint is an equality that can be solved for  $Q_2$ , use the constraint to substitute  $Q_2$  out of the objective function and transform the problem into a problem involving only the choice of  $Q_1$ . Write down the FOC and SOC for an interior local minimum. Write down the meaning of the FOC in standard undergraduate economics language (that is, in words only). Suppose now that two price-taking firms with these cost functions, producing in the same perfectly competitive industry, where they both face the same price of output, collectively produce exactly  $Q^T$  units. Would they also automatically happen to be solving this dictator's problem? Why? What standard undergraduate fact about competitive markets did you just show?

(3) Suppose that a firm uses  $K$  units of capital per year and  $L$  units of labor per year to produce  $Q$  units of output per year. Suppose that the formula output =  $aL^\alpha K^{1-\alpha}$ , where  $\alpha \in (0,1)$  and  $a > 0$ , shows just how much output is produced each year from any combination of amounts of capital and labor per year; such functions are called "production functions" and we'll talk more about them. The partial derivatives of the production function with respect to  $L$  and  $K$  are what you would have called the marginal products of capital and labor, respectively, in your undergraduate courses. The price of a unit of capital for one year is  $r$  and the price of a unit of labor for one year is  $w$ .

Write the requirement that  $K$  and  $L$  must be chosen so as to produce at least  $Q$  units of output as a constraint involving the production function above and the parameter  $Q$ . Consider this the firm's constraint for the problem of minimizing the total cost of producing at least  $Q$  units of output from appropriate choices of capital and labor inputs. Write this down as the problem of minimizing an appropriate objective function subject to that constraint. Argue, by examining this production function, that the firm would never minimize costs by producing more than  $Q$  units, so that we can claim that the constraint is binding. Then solve the constraint for  $K$ , and use the resulting expression to eliminate  $K$  from the objective function, transforming the objective function into a function of  $L$  only. Write out the FOC and SOC for an interior local minimum.

You should remember a condition for a cost-minimizing choice of inputs from your undergraduate days: The marginal product per dollar spent on each input is equivalent for every input. Show that your FOC may be rearranged algebraically so that it is in fact equivalent to that condition.

(4) Consider a monopolist who must choose yearly amounts of capital  $K$  and labor  $L$  to maximize profits. Think of both of these as being rented by the monopolist at the rates  $r$  per unit per year for capital and  $w$  per unit per year for labor. Or if you prefer think of  $r$  as the user cost of a unit of capital each year. Output is produced according to the production function  $Q = f(L,K)$  and the inverse demand for the monopolist's output is  $P = P(Q)$ . Write down the FOCs and SOC's for a profit-maximizing choice of  $L$  and  $K$ .

Algebra-saving hint: It is convenient in this problem to define  $R(Q)$  as total revenue,  $MR(Q)$  as marginal revenue and  $MR'(Q)$  as the first derivative of marginal revenue with respect to  $Q$ , and to write the FOCs and SOC's in terms of these functions, where that is possible (you cannot write them completely in terms of those functions). Otherwise, the FOCs and SOC's are an ugly mess that is hard to interpret; and interpreting SOC's is what the following questions are all about.

(a) For the moment, just assume that the SOC's are satisfied. Show that the FOCs of this profit maximization problem will imply that the monopolist's optimal choices will also satisfy the cost minimization condition that you proved in the previous problem—that the marginal product per dollar spent on each input is identical. In words only, what standard, very basic (but fundamental) undergraduate fact about the relationship between profit maximization and cost minimization have you just proved mathematically?

(b) Now let us return to the SOC's. Suppose now that the production function we are looking at here is  $f(L,K) = L^\alpha K^{1-\alpha}$ , where  $\alpha \in (0,1)$ . What conditions could you place on the derivatives of the inverse demand function to guarantee that the SOC's of the problem are satisfied?

(c) Consider this max problem:

$$\max p(LK)^{0.5} - wL - rK$$

... where  $p$ ,  $w$  and  $r$  are parameters. This is the problem you just looked at with  $P(Q) = p$  and  $\alpha = 0.5$ . Evidently, it is a similar max problem, with a particular value of  $\alpha$ ; it's just that the demand curve is perfectly elastic--in other words, we're considering a price-taking firm (a perfectly competitive firm) here instead of a monopolist. Show that the SOC's for this problem are not satisfied. What is the crucial difference between this problem and the previous monopoly problem, as far as the SOC's go? Explain this difference intuitively in terms of "returns to scale" (look it up in an undergraduate text) and the constancy of the competitive price in the competitive situation. An example (pick a particular  $p$ ,  $w$  and  $r$  in some convenient and useful way and reason about what you get) can be very helpful to illustrate the intuitive problem with the SOC's in this case.

(d) Suppose now that you have the same competitive setup as in part (c), but that the production function in our original setup is instead  $f(L,K) = L^\alpha K^\beta$ , where  $\alpha > 0$ ,  $\beta > 0$ , and  $\alpha + \beta < 1$ . Would you now have to impose the same conditions on  $P(Q)$  to satisfy the SOC's of the problem? Would  $P(Q) = p$  now be ok? Explain again what's going on intuitively in terms of returns to scale and the constancy of the competitive price in the competitive situation. An example (pick a particular  $p$ ,  $w$ ,  $r$ ,  $\alpha$  and  $\beta$ ) can be helpful again. In words, what do your results so far demonstrate about perfect competition and returns to scale?

(5) Recall that in my lecture, I claimed that, for a competitive firm's optimal input demand functions, it was simple to prove (i) cross-price symmetry:

$$\frac{\partial \bar{L}}{\partial r} = \frac{\partial \bar{K}}{\partial w}$$

...but that, in general, we cannot not sign these cross-price effects. I did the formal comparative statics to construct a solution for one of this cross-price effects, but not the other one. Do the comparative statics exercise needed to derive the other one, and combine my result and your result to prove cross-price symmetry by the comparative statics method. (Comment: Do NOT do the cross-price symmetry proof the way I did it in class. That is, don't do it by using the envelope theorem--that is, Hotelling's Lemma---in conjunction with Young's Theorem).